Problem 1. Eigenvalues and Eigenvectors of Matrices

In this problem, we will get our hands dirty with some tedious problems in linear algebra and warm up for the course. You can use R for determining eigenvalues and eigenvectors of the matrix.

Part (a)
What are the eigenvalues of the matrix
\[
A = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2 \\
\end{bmatrix}
\]

Part (b)
Is the matrix A positive definite? Why or why not?

Part (c)
Use R to compute the eigenvectors of the matrix A. Are the eigenvectors linearly independent (an easy way to check if vectors \(e_1, e_2, e_3\) are linearly independent is to show that \(det([e_1 e_2 e_3]) \neq 0\))? 

Part (d)
What is the determinant of the matrix A? What is the product of the eigenvalues of the matrix A? What is the relation between the two numbers?

Part (e)
What is the trace of the matrix A? What is the sum of eigenvalues of the matrix A?

Problem 2. Gram-Schmidt Procedure

We now introduce Gram-Schmidt procedure that is widely used in linear algebra to derive a set of orthogonal vectors. First, we have the following definition.

Definition. Let \(Q\) be a positive definite matrix. A set of vectors \(d_1, \ldots, d_n \in \mathbb{R}^n\) is called mutually \(Q\)-conjugate if \(d_i^T Q d_j = 0\) for all \(i \neq j\).

Now, given positive definite matrix \(Q\) and a set of linearly independent vectors \(\xi_1, \ldots, \xi_n \in \mathbb{R}^n\), how can we construct mutually \(Q\)-conjugate vectors? Gram-Schmidt procedure allows us to do that. Define \(d_1 := \xi_1\). Now, assume that \(d_2 = \xi_2 + c_{21} d_1\), where \(c_{21}\) is a scalar.

Part (a)
Compute the value of \(c_{21}\) such that \(d_1^T Q d_2 = 0\) (one equation, one unknown).

Part (b)
Now assume that \(d_1, \ldots, d_k\) has been computed and are mutually \(Q\)-conjugate. Let \(d_{k+1} = \xi_{k+1} + \sum_{i=1}^{k} c_{k+1,i} d_i\).
Thus, there are \(k\) unknowns here \(c_{k+1,1}, \ldots, c_{k+1,k}\). What are the \(k\) equations in order to compute the unknowns. Compute the unknowns so that \(d_1, \ldots, d_{k+1}\) are mutually \(Q\)-conjugate.
Problem 3. Differentiation of Multivariate Functions

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function of \( n \) variables. The derivative of the function is defined as

\[
\nabla f(x) = \begin{bmatrix}
\nabla_{x_1} f(x) \\
\vdots \\
\nabla_{x_n} f(x)
\end{bmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix}
\nabla^2_{x_1x_1} f(x) & \nabla^2_{x_1x_2} f(x) & \cdots & \nabla^2_{x_1x_n} f(x) \\
\vdots & \ddots & \vdots \\
\nabla^2_{x_nx_1} f(x) & \nabla^2_{x_nx_2} f(x) & \cdots & \nabla^2_{x_nx_n} f(x)
\end{bmatrix}
\]

We define \( f : \mathbb{R}^3 \to \mathbb{R} \) as follows:

\[
f(x) = \sin(x_1x_2) + \cos(x_3).
\]

Find \( \nabla f(x) \) and \( \nabla^2 f(x) \). Is \( \nabla^2 f(x) \) a symmetric matrix?