Note: This preliminary assignment is designed for assessing your background for the CPDA Introduction to Machine Learning class. If you are able to complete this assignment within 1 hour, you have an excellent background for completing Machine Learning class. If you take between 1 hour to 2 hours to complete, then you are likely to struggle in the course. If you take more than 2 hours, then it is recommended that you enroll in the Linear Algebra and Calculus class to brush up on these subjects–they are indispensable for understanding Machine Learning. HELP WILL NOT BE PROVIDED TO COMPLETE THIS ASSIGNMENT.

Problem 1. Eigenvalues and Eigenvectors of Matrices

In this problem, we will get our hands dirty with some tedious problems in linear algebra and warm up for the course. You can use R for determining eigenvalues and eigenvectors of the matrix.

Part (a)

What are the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Part (b)

Is the matrix A positive definite? Why or why not?

Part (c)

Use R to compute the eigenvectors of the matrix A. Are the eigenvectors linearly independent (an easy way to check if vectors e_1, e_2, e_3 are linearly independent is to show that $det([e_1\ e_2\ e_3]) \neq 0$)?

Part (d)

What is the determinant of the matrix A? What is the product of the eigenvalues of the matrix A? What is the relation between the two numbers?

Part (e)

What is the trace of the matrix A? What is the sum of eigenvalues of the matrix A?

Problem 2. Gram-Schmidt Procedure

We now introduce Gram-Schmidt procedure that is widely used in linear algebra to derive a set of orthogonal vectors. First, we have the following definition.

Definition. Let Q be a positive definite matrix. A set of vectors $d_1, \ldots, d_n \in \mathbb{R}^n$ is called **mutually Q-conjugate** if $d_i^T Q d_j = 0$ for all $i \neq j$.

Now, given positive definite matrix Q and a set of linearly independent vectors $\xi_1, \ldots, \xi_n \in \mathbb{R}^n$, how can we construct mutually Q-conjugate vectors? Gram-Schmidt procedure allows us to do that. Define $d_1 := \xi_1$. Now, assume that $d_2 = \xi_2 + c_{21} d_1$, where c_{21} is a scalar.

Part (a)

Compute the value of c_{21} such that $d_1^T Q d_2 = 0$ (one equation, one unknown).

Part (b)

Now assume that $d_1, \ldots, d_k$ has been computed and are mutually Q-conjugate. Let $d_{k+1} = \xi_{k+1} + \sum_{i=1}^{k} c_{k+1,i} d_i$. Thus, there are k unknowns here $c_{k+1,1}, \ldots, c_{k+1,k}$. What are the k equations in order to compute the unknowns. Compute the unknowns so that $d_1, \ldots, d_{k+1}$ are mutually Q-conjugate.

Problem 3. Differentiation of Multivariate Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. The derivative of the function is defined as

$$\nabla f(x) = \begin{bmatrix} \nabla_{x_1} f(x) \\ \vdots \\ \nabla_{x_n} f(x) \end{bmatrix}, \qquad \nabla^2 f(x) = \begin{bmatrix} \nabla^2_{x_1 x_1} f(x) & \nabla^2_{x_1 x_2} f(x) & \dots & \nabla^2_{x_1 x_n} f(x) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^2_{x_n x_1} f(x) & \nabla^2_{x_n x_2} f(x) & \dots & \nabla^2_{x_n x_n} f(x) \end{bmatrix}$$

We define $f : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$f(x) = \sin(x_1 x_2) + \cos(x_3).$$

Find $\nabla f(x)$ and $\nabla^2 f(x)$. Is $\nabla^2 f(x)$ a symmetric matrix?

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